1 Holomorphic motions and λ-lemma:

Holomorphic motion is basically an isotopy of the Riemann sphere parametrized by a complex parameter. The surprising feature of the holomorphic motion is that one does not need to include continuity assumptions in the definition. Analyticity forces strong regularity and extendability properties.

Holomorphic motions were introduced by Mañé-Sad-Sullivan in their seminal paper "On the dynamics of Rational Maps". Many dynamical objects, like Julia sets and Limit Sets of the Kleinian groups, move holomorphically over certain regions in the parameter space. Holomorphic motions also appear in the Teichmuller Theory.

Let $\Delta \subset \mathbb{C}$ a unit disk.

Definition 1.1. *Let $A \subset \bar{\mathbb{C}}$. A holomorphic motion of A is a map $f\colon \Delta \times A \to \bar{\mathbb{C}}$ such that*

1. *for any $a \in A$, the map $\lambda \to f(\lambda, a)$ is holomorphic in Δ*
2. *for any fixed $\lambda \in \Delta$, the map $a \to f(\lambda, a) =: f_\lambda(a)$ is an injection and*
3. *the map f_0 is the identity on A.*

Mañé-Sad-Sullivan proved the following version of λ-Lemma [3]:

Theorem 1.1.
1. *Every holomorphic motion f of $A \subset \mathbb{C}$ extends uniquely to the closure of $\bar{A}$.*
2. *For each λ, the map f_λ is quasisymmetric.*

They asked whether the holomorphic motion can be extended to the holomorphic motion of $\mathbb{C}$. Sullivan and Thurston [5], Bers and Royden [2] showed that there exists a certain δ, so that the holomorphic motion can be extended to the holomorphic motion of $\mathbb{C}$ over a disk of radius δ. Slodkowski showed that δ is actually equal to 1:

Theorem 1.2 (Lambda Lemma,[4])**.** *Each holomorphic motion is a restriction of a holomorphic motion of $\mathbb{C}$.*

Let $T(S)$ be the Teichmuller space modelled on a Riemann surface S. Let $M(S)$ be the space of Beltrami forms. There is a natural projection $\Phi_S : M(S) \to T(S)$.

The Teichmuller Theory reformulation of the λ-Lemma is the following:

Corollary 1.1. *Every holomorphic map* $\gamma : \Delta \to T(S)$ *lifts to a holomorphic map* $\tilde{\gamma} : \Delta \to M(S)$*, such that* $\Phi_S \circ \tilde{\gamma} = \gamma$*.*

There were several attempts to simplify Slodkowski's proof. See Astala-Martin article [1].

We give a new geometric proof of Lambda Lemma. We relate holomorphic motions to the technique of filling totally real manifolds with a boundary in a pseudoconvex domain with holomorphic disks. This technique comes originates from complex analysis and nowadays is extensively used in symplectic geometry. This allows us to replace a major technical part in the Slodkowski's proof by a transparent geometric argument.

[1] K. Astala and G.J. Martin, *Holomorphic motions*, Report. Univ. Jyväskylä **83** (2001), 27-40.

[2] L. Bers and H. Royden, *Holomorphic families of injections*, Acta Math. **157** (1986), 259-286.

[3] R. Mañé, P. Sad, and D. Sullivan, *On the dynamics of rational maps*, Ann. Sci. École Norm. Sup. **16** (1983), 193-217.

[4] Slodkowski, *Holomorphic motions and polynomial hulls*, Proc. Amer. Math Society **111** (1991), 347-355.

[5] D. Sullivan and W. Thurston, *Extending holomorphic motions*, Acta Math **157** (1986), 243-257.